THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XVI. JANUARY, 1909. NO. 1.

A NEW LINKAGE FOR DESCRIBING A STRAIGHT LINE BY CONTINUOUS MOTION.

By JOHN J. QUINN, Ph. D., Pittsburg College, Pittsburg, Pa.

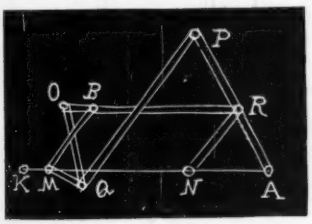
In the linkage given herewith the following conditions obtain: M and N are fixed points to which the whole system is pivoted.

$$MQ=OB=\tfrac{1}{2}OQ;$$

$$OQ=MB=PR=RA=NR=\tfrac{1}{2}QR;$$

$$PQ=OR;\ BR=MN.$$

It is required to find the locus of P and A for every movement of the system if all movable points except P and A describe circles.



1. From the stated conditions, MB is parallel to NR. Hence $\angle OBM=\angle BRN=\angle RNA$. Also $\triangle MOB$ is similar to $\triangle OPR$, since the respective sides are proportional.

Then $\angle OBM=\angle ORP$, being homologous angles of similar triangles. Hence $\angle PRN=2\angle RNA$. Hence it follows that the locus of P is a straight line perpendicular to MN.

2. Since P moves in a straight line, and R is the mid-point of PA, it follows that the locus of A is a straight line. The following corollaries follow immediately:

1. The locus of any point on PA, except R, the mid-point, is an ellipse.

2. On a straight line through MN lay off $MK=MQ$, then K is a fixed point equidistant from the point O.

3. If the link OQ be unfastened at Q, and the point Q fixed at K, the loci of P and A remain unchanged.

THE PERFECT MAGIC SQUARES FOR 1908.*

By DR. G. B. M. ZERR, Philadelphia, Pa.

By Perfect Magic Squares are meant those made up of successive integral numbers. Suppose we denote the number of integers on a side of the Magic Square by n; then, if we start with unity, and write the numbers successively until the square is filled, in magic arrangement, each column, each row, and each of the two diagonals of the square will add to a sum equal to $\frac{n^3+n}{2}$. Whenever $\frac{1908-(n^3+n)/2}{n}$ is an integer we can form a perfect magic square for 1908.

$\frac{1908-(n^3+n)/2}{n}+1=\frac{3816-n^3+n}{2n}$=the least integer permissible for forming the square. $\frac{1908-(n^3+n)/2}{n}+n^2=\frac{3816-n+n^3}{2n}$=the greatest integer permissible for forming the square.

The only values of n for 1908 are 3, 8, and 9.

For $n=3$, least integer=632, greatest integer=640.

For $n=8$, least integer=207, greatest integer=270.

For $n=9$, least integer=172, greatest integer=252.

The three magic squares written out are as follows; all of which sum magically 1908.

$n=3$.

639	632	637
634	636	638
635	640	633

$n=8$.

207	269	209	267	266	212	264	214
262	216	260	218	219	257	221	255
223	253	225	251	250	228	248	230
246	232	244	234	235	241	237	239
238	240	236	242	243	233	245	231
247	229	249	227	226	252	224	254
222	256	220	258	259	217	261	215
263	213	265	211	210	268	208	270

*We pay our respects to the year just gone, through Dr. Zerr's article, and trust that some of our readers may do as much for 1909. EDITORS.

$n=9$.

218	229	240	251	172	183	194	205	216
228	239	250	180	182	193	204	215	217
238	249	179	181	192	203	214	225	227
248	178	189	191	202	213	224	226	237
177	188	190	201	212	223	234	236	247
187	198	200	211	222	238	235	246	176
197	199	210	221	232	243	245	175	186
207	209	220	231	242	244	174	185	196
208	219	230	241	252	173	184	195	206

ON THE EQUATION $x^n+y^n=nz^n$.*

By DR. JACOB WESTLUND, Purdue University.

E. Maillet† has proved that the equation

$$x^n+y^n=nz^n$$

is not solvable in rational integers, when n is a *regular*‡ prime. The object of the following note is to prove the impossibility of the above equation for any odd prime n, when z is not divisible by n.

If z is not divisible by n, it follows that neither x nor y can be divisible by n. We may also, without loss of generality, assume x and y relatively prime. Now it can easily be shown that x^n+y^n is either not divisible by n or is divisible by n^2 at least. For we have

*Read before the Chicago Section of the American Mathematical Society, March, 1908.

†*Annali di Matematica*, Series III, Vol. 12. See also, *Acta Matematica*, Vol. 24.

‡A prime n is said to be a *regular* prime, when it is not a factor of the numerator of any one of the first $(n-3)/2$ Bernoullian numbers.

$$x^n+y^n=(x+y-y)^n+y^n$$
$$=(x+y)^n-n(x+y)^{n-1}y+\ldots+n(x+y)y^{n-1},$$

and since n is an odd prime, it follows that, if x^n+y^n is divisible by n, $x+y$ is also divisible by n, and hence x^n+y^n is divisible by n^2. From this follows directly the impossibility of the equation $x^n+y^n=nz^n$ when z is prime to n.

AN INTERESTING CLASS OF MONOTONIC FUNCTIONS.*

By ARTHUR R. SCHWEITZER, Chicago, Illinois.

1. In the *Rivista di Matematica*, Vol. 2, p. 43, Professor Peano has defined the following monotonic function. Representing the variable x of the interval (0, 1) by the series†

$$x=\sum_{k=1}^{+\infty}\frac{a_k}{m^k} \qquad m\geqq 2,\ a_k\geqq 1,$$

then he defines

$$t(x)=\sum_{k=1}^{+\infty}\left(\frac{a_k}{m^k}\right)^{n_k}$$

where the n_k are integers $\geqq 2$. In this paper the function of Peano is generalized in such a manner that a very large class of monotonic functions is obtained which includes other types than the preceding; also an explicit evaluation of certain properties is given.

2. As is well known,‡ every point $0\leqq x\leqq 1$ can be represented by the series

$$x=\sum_{k=1}^{+\infty}\frac{a_k}{\Pi_k}$$

where

$$\Pi_k=m_1\ m_2\ldots m_k \qquad m_k\geqq 2.$$

The point x has a unique representation when $x=0$ or $x=1$ or $x=r/s$, where r and s are relatively prime and s is not a divisor of the product Π_k for any k. In all other points x is irrational, and therefore has a unique represen-

*In this paper references will be made by notes to the following works: (1) Hobson's *Theory of Functions of a Real Variable*, Cambridge, 1907, (2) Jordan's *Cours d' Analyse*, Vol. I, (3) Chrystal's *Algebra*. These will be briefly referred to as "Hobson," "Jordan," and "Chrystal," respectively.

†Chrystal I, p. 165. Representing X by the radix fraction $\alpha, \alpha_2, \alpha_3\ldots\alpha_n\ldots$ then we obtain $t(x)$ by the suitable interpolation of zeros, *e. g.*, $0\,\alpha_1, 0\,\alpha_2, 0\,\alpha_3\ldots 0\,\alpha_n\ldots$

‡Chrystal, pp. 165, 166.

tation, or x is expressible by a finite number of terms and hence has a double representation. In the latter case x has the repetend zero (0) or the repetend (m_k-1) for an assigned k (m_k-1). Accordingly, we denote the points of the interval (0, 1) by $x=x_f$ and $x=x_\phi$, respectively. If $x=x_f$ and $x=x_\phi$ we define $x=x_\theta$. Thus we have the three sets of representations,

$$X_f=[x_f], \quad X_\phi=[x_\phi], \quad X_\theta=[x_\theta].$$

Hence, in the well known notation of Cantor,*

$$X_\theta = M(X_f, X_\phi),$$

and $\beta(X_f, X_\phi)$ is the set of representations of points admitting expression by an infinite radix fraction only. We denote further

$$R(X_f, X_\phi)=M(X_f, X_\phi)-\beta(X_f, X_\phi),$$

so that $R(X_f, X_\phi)$ is the set of representations of points expressible as a finite radix fraction.

3. We proceed to define the functions $f(x)$, $\phi(x)$, $\theta(x)$. If $x=x_f$ we define

$$f(x)=\sum_{k=1}^{+\infty} rx\frac{a_k^{l_k}}{\Pi'_k}$$

where

$$\Pi'_k=m'_1, m'_2, \ldots, m'_k, \qquad m'_k \geqq 2,$$

the numbers l_k are positive integers $\geqq 1$, and

$$m_k \leqq m'_k, \quad (m_k-1)^{l_k} \leqq m'_k-1;$$

also the numbers r_k are positive. The function $f(x)$ is then monotonic† if the numbers r_k form a non-increasing series. In an analogous manner we

$$f(x)=\sum_{k=1}^{+\infty}\frac{a_k}{(m_1+1)(m_2+1)\ldots(m_k+1)}.$$

define $f(x)$ and $\theta(x)$ corresponding to $x=x_\phi$ and $x=x_\theta$. Thus $f(x)$ and $\phi(x)$ are single-valued, but $\theta(x)$ is not. Denoting

$$F=[f(x)], \quad \Phi=[\phi(x)], \quad \Theta=[\theta(x)],$$

*Hobson, §114; G. Cantor, *Matematische Annalen*, 17, p. 355.

†Hobson, p. 245, §189. In verifying the argument of the text the reader is requested to select from one of the numerous functions defined above a particular example, *e. g.*,

we have

$$\Theta = M(F, \Phi),$$

and corresponding to $\beta(X_f, X_\phi)$ and $R(X_f, X_\phi)$ we have $\beta(F, \Phi)$, $R(F, \Phi)$. As particular cases of the functions $f(x)$, $\phi(x)$, $\theta(x)$ defined above, in addition to the preceding function of Peano, we may mention the functions of G. Cantor, *Matematische Annalen*, 21, p. 590, and Liouville, *Journal de Matematica*, 16, p. 133.

4. In this, and the following sections, we derive explicitly some of the properties of $f(x)$, $\phi(x)$, $\theta(x)$ which we know in advance exist.* The numbers $f(x \pm 0)$, $\phi(x \pm 0)$ are easily evaluated by a suitable selection of sequences. We have, in any point $0 < x_0 < 1$.

$$f(x_0 - 0) = \phi(x_0 - 0),$$
$$f(x_0 + 0) = \phi(x_0 + 0).$$

If x_0 is a point of $R(X_f, X_\phi)$ then

$$f(x_0 - 0) = \phi(x_0),$$
$$f(x_0 + 0) = f(x_0).$$

If x_0 is a point of $\mathfrak{D}(X_f, X_\phi)$ then

$$f(x_0 - 0) = f(x_0),$$
$$f(x_0 + 0) = f(x_0).$$

Thus the functions $f(x)$ and $\phi(x)$ are discontinuous in every point expressible as a finite radix fraction and continuous in every point expressible as an infinite radix fraction only.

5. The set Θ of functional values $\theta(x)$ is nowhere dense and perfect.† Hence the Cantor content‡ of Θ is the difference between the length of the bounding interval and the sum of the point free intervals. For x on the set $R(X_f, X_\phi)$ we have

$$f(x) - \phi(x) = r_\mu \cdot \frac{a_\mu l_\mu - (a_\mu - 1) l_\mu}{\Pi'_\mu} - \sum_{k=1}^{+\infty} r_{\mu+k} \frac{(m_k - 1) l_{\mu+k}}{\Pi'_{\mu+k}},$$

where

$$x = \sum_{k=1}^{\mu} \frac{a_k}{\Pi_k} \qquad a_\mu \neq 0.$$

*Jordan, §71.
†Hobson, §72.
‡Hobson, §85.

Since the number of these intervals is $\Pi_{\mu-1}$, where $\Pi_0=1$, we easily find the sum of the point free intervals to be

$$\sum_{\mu=1}^{+\infty}\left[\frac{\Pi_{\mu-1}}{\Pi'_{\mu-1}}\cdot\sum_{k=0}^{+\infty}r_{\mu+k}\cdot\frac{(m_{\mu+k}-1)l_{\mu+k}}{m'_{\mu}\ldots m'_{\mu+k}}-\frac{\Pi_{\mu}}{\Pi'_{\mu}}\cdot\sum_{k=1}^{+\infty}r_{\mu+k}\frac{(m_{\mu+k}-1)l_{\mu+k}}{m'_{\mu+1}\ldots m'_{\mu+k}}\right]$$

$$=\sum_{k=1}^{+\infty}r_k\cdot\frac{(m_k-1)l_k}{\Pi'_k}-\operatorname*{Limit}_{\mu=+\infty}\frac{\Pi_{\mu}}{\Pi'_{\mu}}\sum_{k=1}^{+\infty}r_{\mu+k}\cdot\frac{(m_{\mu+k}-1)l_{\mu+k}}{m'_{\mu+1}\ldots m'_{\mu+k}}.$$

Hence the content of Θ is

$$C=\operatorname*{Limit}_{\mu=+\infty}\frac{\Pi_{\mu}}{\Pi'_{\mu}}\cdot\sum_{k=1}^{+\infty}r_{\mu+k}\cdot\frac{(m_{\mu+k}-1)l_{\mu+k}}{m'_{\mu+1}\ldots m'_{\mu+k}}.$$

Thus, if $m'_k=m_k^{m_k}$, $m_k\geqq 2$, the corresponding sets Θ have content zero; a similar result is obtained in the case of the Liouville and Cantor functions mentioned above. Also Θ has content zero if the series $r_1+r_2+r_3+$ etc., is convergent. Again, if

$$r_{\mu+k}=1,$$
$$l_{\mu+k}=1,$$
$$m_{\mu+k}=(\mu+k+1)^2-1,$$
$$m'_{\mu+k}=(\mu+k+1)^2,$$

the content of Θ is the infinite product

$$\tfrac{3}{4}\cdot\tfrac{8}{9}\cdot\tfrac{15}{16}\cdots$$

which has a value different from zero.

6. When Θ has content zero we can immediately rectify the curve of the function $f(x)$ in the sense defined by Scheeffer,* *Acta Mathematica*, Vol. 5, p. 59. We require the total discontinuity function of $f(x)$, say $\Omega(x_0, x)$. This function has for $x_0\leqq x\leqq 1$ the value given by the sum of the point free intervals of the subset of Θ contained in the closed interval $[f(x_0), \phi(x)]$, plus the discontinuities of $f(x)$ and $\phi(x)$ in the points x_0, x. We have

*Jordan, p. 90 and following.

$$\Omega(x_0, x)=[\phi(x)-f(x_0)]+[f(x_0)-\phi(x_0)]+[f(x)-\phi(x)]=f(x)-\phi(x_0).$$

Hence $$f(x)-\Omega(x_0, x)=\phi(x_0).$$

Corresponding to the closed interval (x_0, x_1) the length of the function $f(x)-\Omega(x_0, x)$ is x_1-x_0 and hence the length of the curve of $f(x)$ for this interval is

$$L=x_1-x_0+f(x_1)-\phi(x_0).$$

7. Finally, we determine the integral*

$$\int_0^1 f(x)\,dx.$$

We divide the interval (0, 1) into Π_v equal parts. Then every partition point assumes the form

$$x_{.e_1e_2\ldots e_v}=\sum_{k=1}^{v}\frac{e_k}{\Pi_k},$$

and conversely, every radix fraction so expressible is a partition point. Denoting the length of the interval whose first point is $x_{.e_1e_2\ldots e_v}$ by $\delta_{.e_1e_2\ldots e_v}$ we obtain,

$$\int_0^1 f(x)\,dx=\operatorname*{Limit}_{v=+\infty}\sum_{(ek)} f(x_{.e_1e_2\ldots e_v})\ \delta_{.e_1e_2\ldots e_v}$$

$$=\operatorname*{Limit}_{v=+\infty}\sum_{(ek)}\sum_{k=1}^{v} r_k\cdot\frac{e_k^{l_k}}{\Pi'_k}\cdot\frac{1}{\Pi_v}.$$

Now the complete set of partition points (0 included, 1 excluded) may be divided into Π_v / m_k sets, each set being of the type

$$x_{.e_1\ldots e_{k-1}\,0\,e_{k+1}\ldots e_v},\quad x_{.e_1\ldots e_{k-1}\,1\,e_{k+1}\ldots e_v},\ \ldots\ldots\ x_{.e_1\ldots e_{k-1},\,m_k-1,\,e_{k+1}\ldots e_v}.$$

Hence in the complete set of partition points, e_k assumes the sequence of values

$$0, 1, \ldots, m_k-1$$

*Hobson, §225.

precisely Π_v / m_k times. Hence

$$\int_0^1 f(x)\,dx = \underset{v=+\infty}{\text{Limit}} \sum_{k=1}^{v} \frac{r_k}{\Pi_k} \cdot \frac{1}{\Pi_v} \cdot \sum_{(ek)} e_k l_k$$

$$= \underset{v=+\infty}{\text{Limit}} \sum_{k=1}^{v} \frac{r_k}{\Pi_k} \cdot \frac{1}{\Pi_k} \cdot \frac{\Pi_v}{m_k} \cdot \sum_{ek=1}^{m_k-1} e_k l_k$$

$$= \sum_{k=1}^{+\infty} \frac{r_k}{m_k} \cdot \frac{1 l_k + 2 l_k + \ldots + (m_k - 1) l_k}{m'_1\, m'_2 \ldots m'_k}.$$

For example, let

$$1^\circ. \quad l_k = 1,\ r_k = 1,\ m_k = 10,\ m'_k = 10^2.$$

Then

$$\int_0^1 f(x)\,dx = \tfrac{1}{2\,2}.$$

$$2^\circ. \quad l_k = 2,\ r_k = 1,\ m_k = 10,\ m''_k = 10^2.$$

Then

$$\int_0^1 f(x)\,dx = \tfrac{1\,9}{6\,6}.$$

Peano, *loc. cit.*, gives as the values of these integrals $\frac{6}{111}$ and $\frac{32}{111}$, respectively.

8. In conclusion, I wish to thank Professor E. H. Moore for valuable suggestions in the preparation of this paper.

NOTE.—We regret to say that the copy of this article was lost in the mails on its transit to the author. In order to avoid longer delay of this number we undertook the proof reading of the article without the manuscript. Such errors as have escaped will be pointed out in a future issue. ED. F.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Remark on solution of Problem 299, by J. M. ARNOLD, Crompton, R. I.

The following method shows *why* there is only *one* solution, a fact not clearly discernable in the published solution on page 203, Vol. XV.

Let x, y, $3y-x$ be the sides in increasing order of magnitude. Then the area must be $2y-2x$. This leads to an equation of the first degree in x, from which $x=\frac{1}{2}y+\frac{16y}{3y^2+16}$.

Substituting for y, the numbers 1, 2, 3, etc., we find $y=4$, $x=3$. All other integral values of y will give fractional values for x, as the second term soon becomes less than $\frac{1}{2}$, and continues to diminish as y increases.

Hence, 3, 4, 5 are the sides of the only triangle satisfying the problem.

302. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that the system of equations

$$xu+5yv=2,$$
$$xv+yu=1,$$

have no integral solution except one of the unknowns be zero.

I. Solution by E. B. ESCOTT, Ann Arbor, Mich.

Squaring and adding, and multiplying second equation by 5,

$$(xu-5yv)^2+5(xv+yu)^2=9.$$

But, since $(xu-5yv)^2+5(xv+yu)^2=(xu+5yv)^2+5(xv-yu)^2$, we must have the equation

$$X^2+5Y^2=9,$$

satisfied for two sets of values (X, Y).

The only solutions of this last equation are

$$X=\pm 2,\ \pm 3, \quad Y=\pm 1,\ 0,$$

i. e.,

$$xu-5yv=2, \quad xv+yu=1,$$
$$xu+5yv=\pm 3, \quad xv-yu=0.$$

Whence, $2xu=5$ or -1, $2xv=1$, evidently impossible.

Therefore, the only possible solutions are

$$xu-5yv=2, \quad xu+5yv=\pm 2.$$

Whence, $2xu=4$ or 0, $10yv=0$ or -4; *i. e.*, either x, u, y, or v equals zero.

Solved, similarly, by G. B. M. Zerr and V. M. Spunar.

II. Solution by O. C. CARMICHAEL, Oxford, Ala.

If the square of the first equation be added to five times the square of the second equation, we have

$$x^2u^2+5x^2v^2+25y^2v^2+5y^2u^2=9.$$

Therefore, there is no integral solution in x, y, u, v except when one of the unknowns is zero; for, if they were all positive integers, $25y^2v^2$ itself would be greater than 9.

303. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Evaluate the determinant

$$\begin{vmatrix} D_1 & x_1x_2 & x_1x_3 & \dots & x_1x_n \\ x_1x_2 & D_2 & x_2x_3 & \dots & x_2x_n \\ x_1x_3 & x_2x_3 & D_3 & \dots & x_3x_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1x_n & x_2x_n & x_3x_n & \dots & D_n \end{vmatrix}$$

Solution by J. W. CLAWSON, Ursinus College, Collegeville, Pa.

$$\Delta = x_1x_2x_3\dots x_n \begin{vmatrix} D_1/x_1 & x_2 & x_3 & \dots & x_n \\ x_1 & D_2/x_2 & x_3 & \dots & x_n \\ x_1 & x_2 & D_3/x_3 & \dots & x_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1 & x_2 & x_3 & \dots & D_n/x_n \end{vmatrix}$$

$$= \prod_1^n x_1 \begin{vmatrix} (D_1-x_1^2)/x_1 & 0 & 0 & \dots & (x_n^2-D_n)/x_n \\ 0 & (D_2-x_2^2)/x_2 & 0 & \dots & (x_n^2-D_n)/x_n \\ 0 & 0 & (D_3-x_3^2)/x_3 & \dots & (x_n^2-D_n)/x_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1 & x_2 & x_3 & \dots & D_n/x_n \end{vmatrix}$$

Subtracting the last row from each row,

$$=\prod_1^n x_r \cdot \prod_1^n \frac{D_r - x_r^2}{x_r} \begin{vmatrix} 1 & 0 & 0 & \dots & -1 \\ 0 & 1 & 0 & \dots & -1 \\ 0 & 0 & 1 & \dots & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{x_1^2}{D_1 - x_1^2} & \frac{x_2^2}{D_2 - x_2^2} & \frac{x_3^2}{D_3 - x_3^2} & \cdots & \frac{D_n}{D_n - x_n^2} \end{vmatrix}$$

$$=\prod_1^n (D_r - x_r^2) \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & -1 \\ 0 & 0 & 1 & \dots & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{x_1^2}{D_1 - x_1^2} & \frac{x_2^2}{D_2 - x_2^2} & \frac{x_3^2}{D_3 - x_3^2} & \cdots \frac{D_n}{D_n - x_n^2} + \frac{x_1^2}{D_1 - x_1^2} \end{vmatrix}$$

adding last column to the first. Since the first row contains $n-1$ zeros, this reduces the determinant to one of the $(n-1)$st order. Adding last column to the first we reduce to one of the $(n-2)$nd order. Repeat this process $(n-1)$ times in all.

$$\text{Then } \Delta = \prod_1^n (D_r - x_r^2) \left[\frac{D_n}{D_n - x_n^2} + \frac{x_1^2}{D_1 - x_1^2} + \frac{x_2^2}{D_2 - x_2^2} + \dots \frac{x_{n-1}^2}{D_{n-1} - x_{n-1}^2} \right]$$

$$= \prod_1^n (D_r - x_r^2) \left[\frac{x_1^2}{D_1 - x_1^2} + \frac{x_2^2}{D_2 - x_2^2} + \dots \frac{x_{n-1}^2}{D_{n-1} - x_{n-1}^2} + \frac{x_n^2}{D_n - x_n^2} + 1 \right]$$

$$= \prod_1^n (D_r - x_r^2) \left[\sum_1^n \frac{x_r^2}{D_r - x_r^2} + 1 \right].$$

Also solved by G. B. M. Zerr, V. M. Spunar, and J. Scheffer.

304. Proposed by C. N. SCHMALL, New York City.

A policeman on a motor-cycle starts in pursuit of an automobile when the latter has a headway of $\frac{1}{2}$ a mile. A pedestrian who is $\frac{1}{4}$ mile ahead of the auto and who is walking at the rate of 5 miles an hour, notices that when the auto overtakes him the policeman is only 5-12 of a mile behind the auto, and $2\frac{1}{2}$ miles from where the officer started; he overtakes the auto. How long did the chase last?

Solution by G. B. M. ZERR, A. M., Ph. D., and the PROPOSER.

Let x=policeman's rate, y=auto's rate, and z=time for auto to overtake pedestrian.

Then $(5z+\frac{1}{4})/y=(5z+\frac{1}{4}+\frac{1}{12})/x \dots (1)$,

$2/y=(2\frac{1}{2})/x \dots (2)$.

(1)/(2) gives $300z+15=240z+16$. $\therefore z=\frac{1}{60}$ hours=1 minute.

Since it takes the policeman 1 minute to gain $\frac{1}{12}$ mile, it will take him 6 minutes to overtake the auto and end the race.

In the statement of this problem, the semi-colon should be omitted after the word "started." With this omission the problem was solved as above by J. W. Clawson, V. M. Spunar, J. H. Meyer. R. D. Carmichael, A. H. Holmes, B. Kramer, J. Scheffer, J. K. Ellwood, and P. S. Berg, interpreting the problem as printed, agree in the answer being 2 hours, 36 minutes.

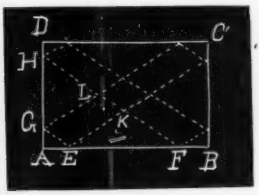
GEOMETRY.

336. Proposed by F. H. HODGE, The University of Chicago.

A man owning a rectangular field $b=300$ feet by $a=600$ feet, wishes to lay out driveways of equal width having the diagonals of the field as center lines and such that the area of the driveways shall be n/m=one-half, of the area of the field. Determine the width of the driveways.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.; NELLIE WOOD, Senior Class, Drury College, Springfield, Mo.; and A. H. HOLMES, Brunswick, Me.

Put $AE=x$; then $AG=\frac{bx}{a}$; $\triangle EFK=\triangle GHL=\frac{b}{a}\cdot\frac{(a-2x)^2}{4}$.



$\therefore$ space occupied by driveways$=ab-\frac{b}{a}(a-2x)^2$.

$$\therefore ab-\frac{b}{a}(a-2x)^2=\frac{n}{m}ab;\text{ whence}$$

$$x=\tfrac{1}{2}a\left(1-\sqrt{1-\frac{n}{m}}\right).$$

$$\therefore\text{breadth of driveway}=\frac{ab}{\sqrt{(a^2+b^2)}}\left(1-\sqrt{1-\frac{n}{m}}\right).$$

For $a=600$, $b=300$, $n:m=1:2$, we find breadth$=60(2\sqrt{5}-\sqrt{10})=78.572$ feet.

Also solved by B. Kramer, V. M. Spunar, G. I. Hopkins, J. H. Meyer, G. B. M. Zerr, and A. H. Bell.

337. Proposed by T. N. HILDEBRANT, The University of Chicago.

Required the locus of the vertices of the parabolae having a given focus and passing through a given point.

Solution by the PROPOSER.

Let O be the given focus and P the given point. From the properties of the parabola we see that the directrices of the parabolae passing through P will be the tangents to the circle of radius OP and center P. Hence the vertices will be the mid-points D of the perpendiculars from O to the tangents. From elementary geometry we have the triangles OCB and OAB equal, and therefore $OC=OA=x_1$ the abscissa of B the point of tangency, if we suppose O the origin of our system of coordinates. Hence $OD=\frac{1}{2}x_1$. Denote by α the angle POC. Then we also have $EPB=\alpha$. Evidently

$$x_1 = OP + PA = a + a\cos\alpha = a(1+\cos\alpha);\ i.\ e.,\ OD = \tfrac{1}{2}x_1 = \frac{a}{2}(1+\cos\alpha),$$

which gives as the polar equation of the locus referred to O as origin and OP as initial line,

$$\rho = \frac{a}{2}(1+\cos\alpha),$$

a cardioid, with cusp at O passing through P and having OP as line of symmetry.

Also solved by J. Scheffer, A. H. Holmes, V. M. Spunar, G. B. M. Zerr, and J. W. Clawson.

CALCULUS.

264. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

The join of the center of curvature of a curve to the origin is at α to the initial line. Prove that with the usual notation:

$$\frac{d\alpha}{d\psi}\left[\left(\frac{dp}{d\psi}\right)^2 + \left(\frac{d^2p}{d\psi^2}\right)^2\right] = \frac{dp}{d\psi}\cdot\frac{d\rho}{d\psi}.$$

Solution by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Denote the coordinates of the center of curvature by ξ, η. Then $\xi = x - \rho\sin\phi$, $\eta = y + \rho\cos\phi$, ρ being the radius of curvature and ϕ the angle formed by the tangent and positive x-axis.

$$\therefore \tan\alpha = \eta/\xi, \text{ and } d\alpha = \frac{\xi\, d\eta - \eta\, d\xi}{\xi^2+\eta^2}.$$

Likewise, $\tan\psi = r\frac{d\theta}{dr}$, and $d\psi = \left[d\theta + r\left(\frac{d^2\theta}{dr^2}\right)\right] \div \sec^2\psi$, where ψ = angle included by the tangent at (r, θ), and the radius vector to the point (r, θ).

Substituting the proper values of ξ, η, $d\xi$, $d\eta$ in $d\alpha$, expressed in polar coordinates, and we have the first product $d\alpha/d\psi$.

Also, on remembering that $\left\{\begin{array}{l} p = r\sin\psi \\ \varphi = \theta + \psi \end{array}\right\}$, differentiate it twice, square every differential quotient as indicated by the proposition, add the like terms, reduce, and we have, readily,

$$\frac{d\alpha}{d\psi}\left[\left(\frac{dp}{d\psi}\right)^2 + \left(\frac{d^2p}{d\psi^2}\right)^2\right] = \frac{dp}{d\psi}\cdot\frac{d\rho}{d\psi}.$$

265. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Find two curves which possess the property that the tangents TP and TQ to the inner one always makes equal angles with the tangent TT' to the outer.

Remarks by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

In Salmon's *Conic Sections*, Sixth Edition, Articles 188-189, it is demonstrated that if through any point P of a conic section we draw tangents PT, QT to a confocal conic section, these tangents are equally inclined to the tangent at P. For every point P the outer conic through P, though confocal to the inner, has different axes.

If we wish to find a curve for all positions of P the problem assumes some difficulty.

For the ellipse, the equation to TP, TQ is

$$(a^2-h^2)(y-k)^2+2(y-k)(x-h)hk+(b^2-k^2)(x-h)^2=0,$$

where (h, k) are the coordinates of T.

$$2hk(y-k)+\{b^2-a^2+h^2-k^2\pm\sqrt{[(b^2-a^2+h^2-k^2)^2+4h^2k^2]}\}(x-h)=0,$$

gives the two lines making equal angles with TP, TQ.

The envelope of the line formed by using the plus sign subject to the condition obtained by using the minus sign gives the required curve.

If $a=b$, the ellipse becomes a circle. Then TT' is

$$ky+hx=h^2+k^2 \ldots (1).$$

$hy=kx$ is the perpendicular to (1).

The envelope of (1) subject to the condition $hy=kx$,is $x^2+y^2=0$, or the center of the given circle.

266. Proposed by C. N. SCHMALL, New York City.

Show that the nth derivative of the fraction u/v can be expressed in the form of a determinant, u and v being functions of x.

Solution by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

As $u=\phi(x)$, $v=\psi(x)$, hence, $F(x)=\phi(x)/\psi(x)=u/v$, and therefore,

$$F'(x)=\frac{vu'-uv'}{v^2}=\frac{1}{v^2}\begin{vmatrix} v & v' \\ u & u' \end{vmatrix}=\frac{u_1}{v_1}=\frac{\phi_1(x)}{\psi_1(x)}.$$

Likewise, $F''(x)=\frac{1}{v_1^2}\begin{vmatrix} v_1 & v'_1 \\ u_1 & u'_1 \end{vmatrix}$; $F'''(x)=\frac{1}{v_2^2}\begin{vmatrix} v_2 & v'_2 \\ u_2 & u'_2 \end{vmatrix}$; ...

$$\therefore F^{(n)}(x)=\frac{1}{v^2{}_{n-1}}\begin{vmatrix} v_{n-1} & v'_{n-1} \\ u_{n-1} & u'_{n-1} \end{vmatrix}, \text{ where } v_\lambda=v^{2^\lambda}, \quad v'_\lambda=2^\lambda v^{2^\lambda-1}v';$$

$$u_\lambda = \begin{vmatrix} v_{\lambda-1} & v'_{\lambda-1} \\ u_{\lambda-1} & u'_{\lambda-1} \end{vmatrix}, \text{ and } u'_\lambda = \begin{vmatrix} v_{\lambda-1} & v''_{\lambda-1} \\ u_{\lambda-1} & u''_{\lambda-1} \end{vmatrix} \ldots$$

Also solved by G. B. M. Zerr.

267. Proposed by FRANK LOXLEY GRIFFIN, S. M., Ph. D., Instructor in Mathematics, Williams College.

A point within an ellipse, upon a normal making an angle λ with the major axis, is arbitrarily chosen. With this point as pole, and the line through it parallel to the major axis as polar axis, the equation of the ellipse is, $A\cos^4\theta+B\cos^3\theta+C\cos^2\theta+D\cos\theta+E=0$, where the coefficients are functions of λ, of the radius vector ρ, and of the distance along the normal to the pole, ρ_1. Evidently for $\rho=\rho_1$, a solution is $\cos\theta=\cos\lambda$. Required the multiplicity of this solution for any values of ρ_1, $[\lambda\neq 0,\ \rho_1\neq 0]$.

Solution by the PROPOSER.

I. *Analytical Solution.* The normal to $\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$ at (x_0, y_0) is, denoting its slope by $\tan\lambda$, $\frac{x_0-\xi}{\cos\lambda}=\frac{y_0-\eta}{\sin\lambda}=r$, where r is the distance from (x_0, y_0) to any point (ξ, η), being positive for interior points. Now, $\tan\lambda=-\frac{dx_0}{dy_0}=\frac{x_0}{x_0(1-e^2)}$, whence $x_0^2+x_0^2(1-e^2)\tan^2\lambda=a^2$, or

$$(1) \qquad x_0=\mu\cos\lambda,\ y_0=\mu\sin\lambda(1-e^2),\ \mu=\frac{a}{\sqrt{(1-e^2\sin^2\lambda)}}.$$

And, if the pole (x_1, y_1):

$$(2) \qquad x_1=x_0-\rho_1\cos\lambda,\ y_1=y_0-\rho_1\sin\lambda.$$

Transfer the origin to (x_1, y_1) and introduce polar coordinates by $x=x_1+\rho\cos\theta$, $y=y_1+\rho\sin\theta$; the ellipse becomes

$$(3) \qquad (1-e^2)(x_1+\rho\cos\theta)^2+(y_1+\rho\sin\theta)^2=a^2(1-e^2).$$

Collect, and eliminate $\sin\theta$:

$$(4) \qquad (\alpha\cos^2\theta+\beta\cos\theta+\gamma)^2=(-\delta\sin\theta)^2=\delta^2(1-\cos^2\theta),$$

$$(5) \qquad \text{where } \alpha=-e^2\rho^2,\ \beta=2x_1\rho(1-e^2),\ \delta=2y_1\rho,$$
$$\text{and } \gamma=\rho^2+(x_1^2-a^2)(1-e^2)+y_1^2.$$

The re-arrangement of (4) in descending powers of $\cos\theta$ gives the form of equation used in stating the problem; the above form will, however, suffice.

When $\rho=\rho_1$ let the coefficients be denoted by $\alpha_1, \ldots, \delta_1$; and let $F(\cos\theta) \equiv (\alpha_1\cos^2\theta+\beta_1\cos\theta+\gamma_1)^2-\delta_1^2(1-\cos^2\theta)$. Now, from the derivative of (4), it is obvious that $\cos\theta=\cos\lambda$ is a solution when $\rho=\rho_1$; *i. e.*, $F(\cos\lambda)=0$. The multiplicity of this solution will be shown by the derivatives. Differentiating with respect to $\cos\theta$, and substituting $\cos\lambda$:

$$(6) \qquad F'(\cos\lambda) \equiv 2(\alpha_1\cos^2\lambda+\beta_1\cos\lambda+\gamma)(2\alpha_1\cos\lambda+\beta_1)+2\delta_1^2\cos\lambda$$
$$=-2\delta_1\sin\lambda(2\alpha_1\cos\lambda+\beta_1)+2\delta_1^2\cos\lambda$$

because of (4). Substituting by (5), (2) and (1):

$$F'(\cos\lambda)=4\delta_1\rho_1\sin\lambda\cos\lambda[e^2\rho_1-(\mu-\rho_1)(1-e^2)+\mu(1-e^2)-\rho_1] \equiv 0.$$

$$(7) \quad \text{Again.} \quad F''(\cos\lambda) \equiv 4\alpha_1(\alpha_1\cos^2\lambda+\beta_1\cos\lambda+\gamma_1)+2(2\alpha_1\cos\lambda+\beta_1)^2+2\delta_1^2$$
$$\equiv -4\alpha_1\delta_1\sin\lambda+2\delta_1^2\cot^2\lambda+2\delta_1^2,$$

because of (4) and (6). Substituting by (5), (2) and (1):

$$(8) \qquad F''(\cos\lambda)=\frac{4\rho_1^2y_1}{\sin\lambda}[\mu(1-e^2)-\rho_1(1-e^2\sin^2\gamma)].$$

This vanishes if, and only if, $y_1=0$ or $\rho_1=\dfrac{\mu(1-e^2)}{1-e^2\sin^2\lambda}$.

$$(9) \qquad \text{Again,} \quad F'''(\cos\lambda) \equiv 12\alpha_1[2\alpha_1\cos\lambda+\beta_1]=\frac{12\alpha_1\delta_1\cos\lambda}{\sin\lambda}.$$

This vanishes if, and only if, $y_1=0$ or $\cos\lambda=0$.

Geometrically, these results show that, with the pole at any point on the normal, the solution $\cos\theta=\cos\lambda$ is of order at least two; with the pole at the center of curvature for the given normal, the solution is of order three; with the pole on the major axis, of order four.

[That $\dfrac{\mu(1-e^2)}{1-e^2\sin^2\lambda}$ is the radius of curvature is seen thus: the usual expression, $R=\dfrac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4}$, reduces immediately by using (1) and $b^2=a^2(1-e^2)$ to $\dfrac{\mu^3(1-e^2)}{a^2}$ or $\dfrac{\mu(1-e^2)}{1-e^2\sin^2\lambda}$.]

The vanishing of $F'''(\cos\lambda)$ when $\cos\lambda=0$ is unimportant, since $F''(\cos\lambda)\neq 0$ when $\cos\lambda=0$.

II. *Geometrical Solution.* Geometrically, $\rho=\rho_1$ is the equation of a circle, of radius ρ_1, with its center at the pole. The order of the solution

$\cos\theta=\cos\lambda$ of the given equation is at least as great as the number of coincident points in which the circle cuts the ellipse at the given point. [How the order may be greater than this number will appear shortly.]

The pole being any point whatever on the normal, the circle is tangent to the ellipse, since the two curves have a common normal. There are consequently at least two coincident points of intersection.

If the pole be at the center of curvature for the given normal, the circle is the osculating circle, which cuts the ellipse in at least three coincident points.

If the pole be on the major axis, then it follows from the symmetry that the circle is tangent to the ellipse at two points, where $\theta=\lambda$ and where $\theta=-\lambda$. But since $\cos(-\lambda)=\cos\lambda$, it follows that for all four intersections $\cos\theta=\cos\lambda$, so that this solution is of multiplicity four.

MECHANICS.

220. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Four particles A, B, C, D, lie on a smooth table at the corners of a rhombus. AB, BC, CD, DA are light inextensible strings connecting the particles. The angle at A is acute. A blow is given to A along the diagonal, away from C. Find the ratio of the initial velocity of C to that of A.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let v=velocity of A. Then since the table and particles are smooth the component of v along the direction BA is $y=v\cos\frac{1}{2}A$. The component of y in the direction CB is $z=y\cos A=v\cos A\cos\frac{1}{2}A$. The component of z in the direction CA is $u=z\cos\frac{1}{2}A$.

$\therefore u=v\cos A\cos^2\frac{1}{2}A$. But u is the initial velocity of C.

$\therefore v : u=1 : \cos A\cos^2\frac{1}{2}A$.

As $\angle A$ decreases every instant, u increases every instant until $\angle A=0°$, $u=v$.

221. Proposed by W. J. GREENSTREET, Stroud, England.

Two smooth intersecting planes are each at 45° to the horizon. Between them lies a cylinder of elliptic cross section. Find the position of equilibrium.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let P, Q be the points of contact of the cylinder with the planes, intersecting at an angle of 90°. Let the normals at P, Q intersect in O and let C be the center of an elliptic section in the same plane as OP, OQ. Then either C and O coincide or CO is vertical. If they coincide the axes of the elliptic section are parallel to the planes. This gives one position of equilibrium. When OC is vertical, OC makes an angle of 45° with both planes.

From the force diagram the reactions at P, Q are equal, hence either the major axis is vertical or the minor axis is vertical, giving the other two positions of equilibrium.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

156. Proposed by A. H. HOLMES, Brunswick, Maine.

Find integral values for a, b, c, d, and e in the equation, $a^2+b^2+c^2+d^2=e^2$.

I. Solution by the PROPOSER.

Take $a=32(40^2-9^2)$; $b=64(40\times9)$; $c=24(40^2+9^2)$; $d=9(40^2+9^2)$. Then $a^2=32^2\times1519^2$; $b^2=32^2\times720^2$; $c^2=24^2\times1681^2$; $d^2=81\times1681^2$. And $e^3=32^2[1519^2+720^2+1681^2(24^2+81)]$.

$\therefore a=48608$; $b=23040$; $c=40344$; $d=15129$; $e=1681$.

In the above problem it was intended by the author that e^2 should be e^3. ED. F.

II. Solution by ARTEMAS MARTIN, LL. D., Coast Survey Office, Washington, D. C.

In the well-known identity

$$(x-y)^2+4xy=(x+y)^2, \tag{1}$$

x and y may have any values whatever, and we can, therefore, assume $x=u+v+w$; then (1) becomes

$$(u+v+w-y)^2+4y(u+v+w)=(u+v+w+y)^2. \tag{2}$$

Now take $u=p^2$, $v=q^2$, $w=r^2$, $y=s^2$; then (2) becomes

$$(p^2+q^2+r^2-s^2)^2+(2ps)^2+(2qs)^2+(2rs)^2=(p^2+q^2+r^2+s^2)^2, \tag{3};$$

and p, q, r, s may be any values chosen at pleasure.

Therefore we may take $a=p^2+q^2+r^2+s^2$, $b=2ps$, $c=2qs$, $d=2rs$, $e=p^2+q^2+r^2+s^2$, or in any other order we please.

Examples. 1. Take $p=1$, $q=2$, $r=3$, $s=1$; then $2^2+4^2+6^2+13^2=15^2$.

2. Take $p=1$, $q=2$, $r=3$, $s=4$; then, after dividing through by 2^2 and discarding the negative sign, $1^2+4^2+8^2+12^2=15^2$.

3. Take $p=2$, $q=3$, $r=4$, $s=5$; then, after dividing through by 2^2, $2^2+10^2+15^2+20^2=27^2$.

See *Mathematical Magazine*, Vol. II, No. 5, pp. 69-76; No. 6, pp. 89-96; No. 8, pp. 137-140; and No. 11, pp. 209-220, for various methods of finding many sets of square numbers whose sum is a square.

III. Solution by ARTEMAS MARTIN, LL. D., Coast Survey Office, Washington, D. C.

Let $a=x+p$, $b=x-p$, $c=x+q$, $d=x-q$, $e=2x+n$; then by substitution and reduction, $4x^2+2p^2+2q^2$, from which

$$x=\frac{2p^2+2q^2-n^2}{4n}.$$

Substituting the value of x and rejecting the common denominator $4n$, we may take

$$a=2p^2+2q^2-n^2+4np,$$
$$b=2p^2+2q^2-n^2-4np,$$
$$c=2p^2+2q^2-n^2+4nq,$$
$$d=2p^2+2q^2-n^2-4nq,$$
$$e=4p^2+4q^2+2n^2.$$

Examples. 1. Take $p=3$, $q=2$, $n=1$; then we have $13^2+17^2+33^2+37^2=54^2$.

2. Take $p=1$, $q=2$, $n=3$, and we have $11^2+13^2+23^2+25^2=38^2$.

3. Take $p=3$, $q=4$, $n=2$, and we get $7^2+11^2+35^2+39^2=54^2$.

Also solved by V. M. Spunar, S. E. Corey, G. B. M. Zerr, and J. Scheffer.

NOTES AND NEWS.

This first number of the sixteenth volume of the MONTHLY has been delayed for various reasons, including the culmination of plans whereby its future publication is to be under the joint auspices of The University of Chicago and the University of Illinois. In accordance with this plan, Professor G. A. Miller, of the University of Illinois, and Professor H. E. Slaught, of the University of Chicago, will share jointly the editorial responsibility of articles. Professor B. F. Finkel, of Drury College, will continue in charge of the problem department and have general management of the MONTHLY, as heretofore, and Professor L. E. Dickson, of The University of Chicago, whose editorial cooperation has been of the highest value, will retire from active service. It is believed that the union of these two universities in the interests of the MONTHLY will result in the further development of its usefulness and in the extension of its influence in its particular field, namely, the realm of college mathematics as distinguished, on the one hand, from that of the secondary schools, and on the other hand, from that of the graduate schools.

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THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XVI. FEBRUARY, 1909. NO. 2.

SOME NOTES ON VECTOR ANALYSIS.*

By ARTHUR C. LUNN, The University of Chicago.

1. The Parametric Representation of a Dyadic of Rotation.

If a rigid body undergo any displacement or change of orientation, keeping one point fixed, the rectangular coordinates of any point after the motion, referred to the fixed point as origin, are expressible in terms of the coordinates of the same point before the motion by the equations

$$\begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z, \\ y' &= a_{21}x + a_{22}y + a_{23}z, \\ z' &= a_{31}x + a_{32}y + a_{33}z, \end{aligned} \tag{1}$$

where the coefficients a_{ij} constitute an orthogonal matrix or dyadic, being subject to the conditions

$$\begin{aligned} a_{11}^2 + a_{21}^2 + a_{31}^2 &= 1, \quad a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0, \\ a_{12}^2 + a_{22}^2 + a_{32}^2 &= 1, \quad a_{13}a_{11} + a_{23}a_{21} + a_{33}a_{31} = 0, \\ a_{13}^2 + a_{23}^2 + a_{33}^2 &= 1, \quad a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0, \end{aligned} \tag{2}$$

which are obvious geometrically from the interpretation of the a's, and of these combinations of them, as cosines of angles formed by various pairs of lines; or algebraically from the fact that the relation $x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2$ must be an identity in x, y, z. These relations are independent and sufficient, so that three of the coefficients are arbitrary; or, more generally, the nine coefficients may be considered as functions of three arbitrary parameters.

Euler showed that any such displacement could be produced by a single rotation through a certain angle about a determinate axis, and therefore that the coefficients could be expressed in terms of the angle of rotation t

*For the notation and theorems of vector analysis here used, see for example: Gibbs-Wilson, *Vector Analysis*, to which reference is made in following foot-notes.